Educational and Psychological Measurement
1983, 43

A RECIPROCITY PRINCIPLE THAT PERMITS PROFILE-FACTORING WITH VERY LARGE NUMBERS OF PEOPLE[1]

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Analytical methods of factoring score profiles are impractical to apply to large cross-products matrices of profile similarities. Basic matrix relationships are presented and illustrated which permit computation of loadings of people on factors from the factor analysis of the usually smaller variable by variable cross-products matrix. Some important additional issues are discussed and mathematical relationships are stated, particularly those concerning the rotation of factor axes.

A persistent issue in psychometrics has been that of developing appropriate methods of multivariate analysis for the grouping of score profiles into types, clusters, or factors. Such score profiles consist of the scores of n people on any m measurements. Examples would be the scores of 800 mental-hospital patients on the nine scales of the MMPI or the scores of 400 male college students on 16 physiological measures concerning responses to various types of exercise-induced stress.

In the early 1950s it was suggested that it would be useful to view the profile-factoring problem in terms of the euclidean distance measure D, which is the square root of the sum of squared differences in scores on the elements of any two score profiles. This suggestion was made by Osgood and Suci (1952), and the D measure and related statistics were discussed fully by Cronbach and Gleser (1953). As a way of portraying similarities and differences among profiles, the D measure made good sense; however, the powerful methods of multivariate analysis needed to derive profile factors could not be applied to the D measure.

[1] Financial support for this work was provided in part by a grant from the National Institute of Mental Health No. 1-R01-MH-35720-01.

Nunnally (1962) demonstrated that all of the information obtainable from the *D* measure also was obtainable from the raw score cross-products between profiles. Score profiles could be dimensionalized by factoring the $n \times n$ matrix of cross-products between profiles. The history of this problem and a detailed discussion of "raw score factor analysis" are given in Nunnally (1978, chapter 12).

The computational method for factoring profiles discussed in this paper is based on the equivalence of solutions obtained from profile factoring and factoring the $m \times m$ matrix of cross-products between measures, which will subsequently be called a variable analysis. This reciprocity principle is implicit in the Eckart and Young (1936) theorem and was discussed in the psychological literature in a series of articles ranging from those of Burt (1937), Stephenson (1936), and Sandler (1949), up to more recent literature including Broverman (1961, 1963), Gollob (1968a, 1968b, 1968c), MacAndrew and Forgy (1963), Ross (1963), Ryder (1964) Skinner (1978), and Tucker (1968). However, for three reasons we feel that the present article is necessary. First, from comparing the various statements made by authors who have discussed aspects and implications of the reciprocity principle, we detect considerable misunderstanding and controversy on the matter. Second, the reciprocity principle has been discussed almost wholly in the context of disputes as to what if anything one loses or gains by employing a particular approach to statistically preparing scores before they are submitted to factor analysis, as in the longstanding dispute that raged over whether Q-technique as an approach to factoring really provided anything different from the more customary R-technique. Aside from these arguments concerning the usefulness of different factor-analytic designs, the reciprocity principle is entirely legitimate in its own right and can be used to effect a very large savings in computational effort in the analysis of profile data. Third, there are some important issues and problems that arise in relation to the reciprocity principle which have never been explained either in words or in terms of mathematical relationships, this particularly being the case for the important issue of rotating factor axes. It will be attempted in this paper to clarify these three matters, to provide appropriate mathematical developments, and to illustrate the issues and proposed solutions concretely.

Some Principles of Profile Analysis

In employing factor analysis, psychologists are accustomed to thinking in terms of correlation coefficients. With respect to the problem of profile factoring, this would be conducted as follows.

First, raw-scores would be stardardized separately for each person in each column of the score matrix S. Then the well-known theorem would hold:

$$\mathbf{R} = 1/m\ \mathbf{S'S} \tag{1}$$

where $\mathbf{R}$ = a $n \times n$ symmetric matrix of correlation coefficients among the profiles for the n people, m = number of measures obtained in the profile, and $\mathbf{S}$ = the score matrix. Of course, such a correlation matrix can be subjected to any and all methods of factor analysis, rotation, and a wide variety of multivariate correlational methods. One of the early proposals was to investigate similarities among profile *shapes* by factoring $\mathbf{R}$; but as was pointed out by Cronbach and Gleser (1953), that would throw away potentially useful information concerning profile *level* and *dispersion*. The profile level relates to the mean score of the individual over the m profile elements, while dispersion relates to the variability among the profile elements for an individual. It is possible to preserve the information relating to profile dispersion by factoring the covariances among the profiles for individuals rather than the correlations. However, the analysis of such convariances also excludes all evidence relating to profile level.

It was demonstrated by Nunnally (1962) that all information in a profile (including shape, level, and dispersion) can be investigated by factor analyzing the sums of cross products among scores on profiles when neither the means within individual profiles are equated to zero (as in covariance analysis) or scores are standardized (as in correlation analysis). Given the legitimacy of factoring any matrix of cross products, it was recommended that one simply factor analyze the matrix $\mathbf{S'S}$, with $\mathbf{S}$ in any form that would be most interpretable. This general approach to factoring allows one to derive dimensionalized spaces of score profiles from scores in any form—raw, mean deviate, or standardized. Nunnally (1978) discussed the mathematical logic of these approaches and the various methods that can be employed to transform score profiles within individuals or over persons on profile elements prior to computing the matrix of cross-products.

The Practical Problem

From the foregoing discussion it might be concluded that the major issues in the multivariate analysis of profile factoring (clustering or typing) have been settled. Unfortunately, a major practical problem exists which has limited the usefulness of methods of profile factoring: the unmanageable size of matrices that might be analyzed in large-scale problems. The size of the cross-product matrix of score profiles depends upon the number of people in the investigation rather than the number of measures or profile elements. Thus, the previously given example of investigating profile factors among 800 mental-hospital patients would require the investigator to compute a 800 × 800 matrix and subject that matrix to some form of factor analysis, e.g., principal-component analysis followed by analytical methods of rotation. It becomes obvious that restrictions on available computer work space, computer time, and related software prohibit the analysis of major problems in profile analysis. Also, even if it were possible to perform the analysis, the investigator might become quite lost in studying the thousands of factor loadings and other products of the analyses.

There is a way of analyzing large-scale problems in profile analysis which demands far less of computer resources and also probably provides solutions that are more easily interpretable by investigators. This alternative method for obtaining factor loadings for profiles can be developed easily from the Eckart-Young (1936) theorem, which forms the mathematical basis for one of the fundamental theorems of factor analysis. Any matrix of real numbers can be factored as follows:

$$\mathbf{S} = \mathbf{FP} \tag{2}$$

where **F** = a matrix of factor loadings with the number of columns equal to the rank of **S**. **P** consists of factor scores, with as many rows as there are factors. A more general expression is as follows:

$$\mathbf{S} = \mathbf{VDP} \tag{3}$$

In the above expression, the columns of **F** are converted to "unit length" by dividing each element in each column by the square root of the sum of squared elements in a column. The values required to normalize **F** in this way are then shown in a diagonal matrix **D**. The elements of **D** are standard deviations when dealing with correlational analysis or covariance analysis, and are analogous to such standard deviations in raw score factoring. Whereas the formulation of factor analysis in equation 3 applies to all approaches to factoring including rotated solutions, it is most frequently shown in that form with respect to principal-components factoring. In that case, **V** is orthogonal by columns, and thus $\mathbf{V'V}$ = an identity matrix **I**. If **V** is of "full rank," that is if there are as many columns in **V** as the rank of **S**, then **V** is orthogonal by rows also, and thus $\mathbf{VV'} = \mathbf{I}$. By definition, **P** is orthogonal by rows and derived in such a way as to be normalized by rows. Thus, $\mathbf{PP'} = \mathbf{I}$. Also, in a full-rank factoring, $\mathbf{P'P} = \mathbf{I}$. As the problem of factoring is formulated in Eq. 3, the elements of **D** may either be combined with **V** to produce the usual-appearing set of factor loadings, or it can be multiplied into **P** in such a way as to express factor scores in terms of the standard deviations (or vector lengths) corresponding to the given factors.

In profile analysis, matrices **V**, **D**, and **P** are obtained from the factor analysis of the profile cross-products matrix **S′S**. However, the same matrices **V**, **D**, and **P** can be obtained by factoring the variable cross-products matrix **SS′**. There is a very simple relationship between the two factor solutions: the **V** matrix in one case is the **P′** matrix in the other case, and the **D** matrix, containing the square roots of the eigenvalues, remains the same. This being the case, it makes no difference in the calculations required for principal components factor analysis whether a profile analysis or a variable analysis is conducted.

Obviously, if the difference between the number of profile and variable elements is large, factoring the smaller cross-products matrix will result in large savings of computational effort as well as "mental strain" in trying to decipher the much larger number of factor loadings that would be obtained from factoring the larger matrix. In the example of performing profile analysis with 800 patients on 9 MMPI scales, one need only factor analyze a 9 × 9 matrix rather than 800 × 800 matrix. Because this reciprocity principle has rarely been applied in the literature on profile analysis (as far as the authors know), its use will be illustrated, and some principles regarding related methods of analysis will be discussed.

An Illustration

The reciprocity principle is demonstrated using the data matrix in Table 1, which depicts the scores of six persons on four variables. This **S** matrix has a rank of 2. Profile analysis could be undertaken by forming **S′S**. The resulting matrix of profile cross products is shown in Table 2 along with a full-rank factoring by principal

TABLE 1
Raw Score Matrix S

	Person					
Measure	A	B	C	D	E	F
1	8	4	2	4	2	1
2	4	2	1	8	4	2
3	4	2	1	8	4	2
4	8	4	2	4	2	1

TABLE 2

Full Principal Components Analysis of the Person × Person Matrix

	SS′ Person						=	V Factor	
Person	A	B	C	D	E	F	Person	I	II
A	160	80	40	128	64	32	A	.617	−.617
B	80	40	20	64	32	16	B	.309	−.309
C	40	20	10	32	16	8	C	.154	−.154
D	128	64	32	160	80	40	D	.167	.617
E	64	32	16	80	40	20	E	.309	.309
F	32	16	8	40	20	10	F	.154	.154
							Eigenvalue	378	42

components into the matrices **V**, **D**, and **P**. (Whereas the numbers in **V**, **D**, and **P** may be difficult to interpret by persons who are accustomed to dealing with correlation matrices rather than matrices of cross-product terms more generally, the elements in these matrices can be made more interpretable by various statistical treatments, such as transforming the elements within columns of **V** into standard scores, transformed standard scores, or into percentiles).

As a computational alternative to the profile analysis shown in Table 2, one can perform a variable analysis using **SS′**. The matrix of variable cross products is shown in Table 3 (a 4 × 4 rather than a 6 × 6 matrix) along with a principal components factoring into the matrices **V**, **D**, and **P**. It is obvious that the matrix **D** in Table 2 contains the same two positive elements (square roots of eigenvalues) as in Table 3. It is also obvious that the **V** matrix in one solution equals the **P′** matrix in the other solution and vice versa. Thus, to obtain eigenvalues, normalized factor loadings, and factor scores, it makes no difference which approach is employed. Obviously, it behooves the investigator to perform the analysis on the smaller of the two cross-products matrices.

TABLE 3

Full Principal Components Analysis of the Measure × Measure Matrix

	S′S Measure				=	V Factor		
Measure	1	2	3	4	Measure	I	II	Factor
1	105	84	84	105	1	.5	−.5	I
2	84	105	105	84	2	.5	.5	II
3	84	105	105	84	3	.5	.5	
4	105	84	84	105	4	.5	−.5	
					Eigenvalue	378	42	

	D Factor			P Measure			
Factor	I	II	Factor	1	2	3	4
I	19.4	0	I	.5	.5	.5	.5
II	0	6.4	II	−.5	.5	.5	−.5

Rotation of Factors

In Tables 1–3, the reciprocity principle was illustrated with principal components factoring before any rotations were made. In profile factoring, as in all factoring, the investigator usually would want to rotate the factors, and for that purpose all the commonly employed analytical computerized routines can be employed with respect to the sets of factor loadings. Thus, in an exploratory factor analysis of score profiles, one might want to employ the Varimax method of computerized rotation to obtain an orthogonal approximation of "simple structure." This would tend to isolate some individuals as being "pure types" with respect to each of the factors, which probably would facilitate interpretation of the factors and represent interesting findings at that stage of the research.

The reciprocity principle holds when an orthogonal transformation matrix **T** is used to rotate the factor loading matrix to a more interpretable solution:

$$\mathbf{S} = \mathbf{FTT'P} \tag{4}$$

For example, the varimax solution to **F**, obtained from a principal

D Factor			P Person					
I	II	Factor	A	B	C	D	E	F
19.4	0	I	.617	.309	.154	.617	.309	.154
0	6.4	II	−.617	−.309	−.154	.617	.309	.154

components analysis of variables, can be partitioned into $\mathbf{V}_r\mathbf{D}_r$. The $\mathbf{D}_r$ matrix no longer consists of the square roots of the eigenvalues but rather contains the square roots of the sums of squared elements in the columns of the rotated factor loading matrix $\mathbf{F}_r$. Premultiplication of $\mathbf{D}_r$ into $\mathbf{T}'\mathbf{P}$ results in the same factor loadings obtained from postmultiplying the principal components found in a profile analysis by $\mathbf{T}$.

Unfortunately, in making the transformation discussed above, one necessarily forces the basis of factor scores to be nonorthogonal. After this transformation is made, the normalized columns of the $\mathbf{F}_r$ matrix would become the factor scores corresponding to the loadings for people on the profile factors. As is generally known, such rotated factor matrices are not orthogonal by columns as are unrotated principal component loadings. One can easily obtain the cosines among the columns of the $\mathbf{F}_r$ matrix by taking $\mathbf{V}'_r\mathbf{V}_r$. Apparently there is no way out of the abandonment of orthogonal factors when employing the reciprocity principle after rotation of $\mathbf{F}$. Either the investigator accepts the sizes of the cosines among factor axes as being reasonable, or instead he rotates the $\mathbf{DP}$ matrix rather than the $\mathbf{F}$ matrix as will be described subsequently.

Working with the reciprocity principal leads one to conceptualize factor analysis as a system made up of two matrices (plus a diagonal matrix of scale factors) each of which can be thought of alternatively as being factor loadings or factor scores depending on whether the diagonal matrix of scale factors is multiplied into one or the other. A conceptual problem arises as to how one should represent the system. If one is seeking a structure of factor loadings, it is an open question as to whether it would be more helpful to interpret the results in relation to the person-person defined space of factors or the measurement-measurement defined space of factors. One might argue that it is sensible to place the measurement basis for the profiles in an approximation of simple structure in the matrix of factor loadings, make interpretations accordingly, and then simply index the profile factor results for people in relation to that simple structure rather than in terms of a simple structure on the people-loadings. This would be accomplished in a variable analysis by factoring $\mathbf{SS}'$ to obtain $\mathbf{F}$ and $\mathbf{P}$, rotating $\mathbf{F}$ by obtaining the transformation matrix $\mathbf{T}$, computing $\mathbf{D}$ from $\mathbf{FT}$, and computing $\mathbf{D}_r\mathbf{T}'\mathbf{P}$. However, a mathematically legitimate alternative approach would be to employ the analytical methods of rotation (e.g., Varimax) on the $\mathbf{P}'\mathbf{D}$ matrix. Thus, in the previous example of 9 MMPI scales and 800 subjects, if four profile factors were derived from the 9×9 analysis of the profile measurements, one could subsequently apply the computerized method of rotation to the 800 × 4 matrix of factor loadings in **P′D**. When that approach is used, the people-loadings are with respect to an orthogonal basis in $\mathbf{V}_r$. As an additional possibility, if the number of people in the investigation is huge, one can randomly sample from the total group of subjects, obtain the necessary Varimax transformation matrix from the sample, and then subsequently apply that transformation to all individuals. This can be done easily, because once the transformation matrix is obtained, it can be employed successively on subjects rather than with all subjects considered simultaneously.

The Common Factor Model

A final issue is that of the frequently-employed practice of inserting "communality estimates" in the diagonals of correlation matrices before factoring. This approach is employed in the search for "common factors" as distinct from the forms of component analysis that have been discussed in this paper. Interestingly, the possibilities of such common factor analysis in relation to the profile factor problem have never been seriously discussed—at least so far as we are aware. As it has been discussed here, the reciprocity principle applies only to component analysis. If any form of mathematical reciprocity holds in the case of common factor analysis, we are not aware of it.

Summary and Conclusions

One of the major drawbacks to applying methods of component factor analysis to the study of profile types or clusters is that available methods of analysis would require unbearable amounts of computations in problems where more than several hundred subjects are involved. Even where the number of subjects is not beyond the capacity of available computers (e.g., 150), the products of such analyses might defy easy interpretation. A reciprocity principle was presented which greatly simplifies the computation of factor loadings and other products of profile factoring, even when undertaken on thousands of subjects. Also, the use of the reciprocity principle might help simplify the interpretation of results.

In psychology, one continually hears mention of the possible value of investigating profile factors or types (e.g., with respect to classifications of the mentally ill or with respect to patterns of physiological response to stress); however, there appears to be a great deal of confusion about the logic and technique of profile analysis, and apparently very little actual research in that regard has appeared in the literature. It is hoped that the reciprocity principle discussed and illustrated in this article will make it much simpler to perform profile analyses in future research, and it is hoped also that the illustrations in this article of profile analysis at work will serve to familiarize readers with a potentially valuable approach to research on human characteristics and powerful, easily-applied methods of mathematical analysis.

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